Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **78**, 1–22; https://doi.org/10.14232/ejqtde.2017.1.78 www.math.u-szeged.hu/ejqtde/

Strong and weak admissibility of L^∞ spaces

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Received 27 June 2017, appeared 17 November 2017
Communicated by Christian Pötzsche

Abstract. For a dynamics with continuous time, we consider the notion of a *strong* exponential dichotomy with respect to a family of norms and we characterize it in terms of the admissibility of bounded solutions. Moreover, we consider both strong and weak admissibility, in the sense that the solutions are respectively of a nonautonomous linear equation defined by a strongly continuous function or of an integral equation obtained from perturbing a general evolution family. As a nontrivial application, we establish the robustness of the notions of a strong exponential dichotomy and of a strong *nonuniform* exponential dichotomy. We emphasize that the last notion is ubiquitous in the context of ergodic theory: for almost all trajectories with nonzero Lyapunov exponents of a measure-preserving flow, the linear variational equation admits a strong nonuniform exponential dichotomy..

Keywords: admissibility, continuous time, exponential dichotomies.

2010 Mathematics Subject Classification: 37D99.

1 Introduction

For a nonautonomous linear equation

$$x' = A(t)x \tag{1.1}$$

in a Banach space defined by a strongly continuous function $A(t)$ and more generally for an evolution family $T(t,s)$ in a Banach space, we introduce the notion of a strong exponential dichotomy with respect to a family of norms. This means that besides having the usual upper bounds in the stable direction for positive time and in the unstable direction for negative time, we have, in addition, lower bounds in the stable direction for positive time and in the unstable direction for negative time.

Moreover, at each time we consider a possibly different norm. The main motivation comes from ergodic theory. Indeed, for almost all trajectories with nonzero Lyapunov exponents of a measure-preserving flow, the linear variational equation admits a strong nonuniform exponential dichotomy (we refer to [2] for details and references). This last notion is a particular

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case of the notion of a strong exponential dichotomy with respect to a family of norms, more precisely a family of Lyapunov norms. Therefore, the type of exponential behavior considered in the paper, besides being very common in the context of ergodic theory, plays a unifying role. In particular, it includes as particular cases both the notions of uniform and nonuniform exponential behavior, considering respectively families of constant norms and Lyapunov norms.

Our main aim is to characterize the notion of a strong exponential dichotomy in terms of the admissibility of bounded solutions. The latter corresponds to assume that there exists a unique bounded solution for each time-dependent bounded perturbation of the original dynamics. In addition to considering a nonautonomous linear equation and more generally an arbitrary evolution family, we also consider both strong and weak admissibility, which corresponds to the perturbations of each of those dynamics. More precisely, in the case of equation (1.1) we consider the perturbed equation

$$x' = A(t)x + y(t) \tag{1.2}$$

and its classical solutions, while in the case of an arbitrary evolution family $T(t,s)$ we consider the perturbed integral equation

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \tag{1.3}$$

and its mild solutions. We refer to the admissibility in the two perturbed equations, respectively, as *strong* and *weak admissibility*. We emphasize that a priori none of them implies the other.

Our main results show that:

1. the evolution family defined by equation (1.1) admits a *strong* exponential dichotomy with respect to a family of norms if and only if it has bounded growth and there exists a unique bounded solution of equation (1.2) for each bounded perturbation y of the original dynamics (see Theorems 2.1 and 2.3);

2. an arbitrary evolution family $T(t,s)$ admits a *strong* exponential dichotomy with respect to a family of norms if and only if it has bounded growth and there exists a unique bounded solution of equation (1.3) for each bounded perturbation y of the original dynamics (see Theorems 4.1 and 4.2).

Here, "bounded growth" and "bounded" are always with respect to the family of norms $\|\cdot\|_t$ under consideration. For example, a function $y\colon \mathbb{R} \to X$ with values in a Banach space X is said to be *bounded* (with respect to the norms $\|\cdot\|_t$) if

$$\sup_{t\in\mathbb{R}} \|y(t)\|_t < +\infty.$$

For an evolution family with bounded growth defined by a differential equation as in (1.1), it follows from the latter results that there exists a unique bounded solution of equation (1.2) for each bounded perturbation y if and only if there exists a unique bounded solution of equation (1.3) for each bounded perturbation y. In other words, in our setting the notions of weak admissibility and strong admissibility are in fact equivalent. In fact, this can be considered the main contribution of our work.

The study of the admissibility property goes back to pioneering work of Perron in [8] who used it to deduce the stability or the conditional stability under sufficiently small perturbations of a linear equation. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [6] and by Dalec′kiĭ and Kreĭn [4]. We also refer to [5] for some early results in infinite-dimensional spaces.

As a nontrivial application of these results, we establish the robustness of the notion of a strong exponential dichotomy with respect to a family of norms and of a strong *nonuniform* exponential dichotomy. This corresponds to show that any sufficiently small linear perturbation of the dynamics is still, respectively, a strong exponential dichotomy with respect to a family of norms and a strong nonuniform exponential dichotomy. We emphasize that the study of robustness has a long history; see in particular [3,7,9,10] and the references therein. See also [1] for the study of robustness in the general setting of a nonuniform exponential behavior.

2 Exponential behavior and strong admissibility

2.1 Exponential dichotomies

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators on X. A function $A\colon \mathbb{R} \to B(X)$ is said to be *strongly continuous* if for each $x \in X$ the map $t \mapsto A(t)x$ is continuous. We note that every continuous function $A\colon \mathbb{R} \to B(X)$ is strongly continuous.

Let $A\colon \mathbb{R} \to B(X)$ be a strongly continuous function and consider the linear equation

$$x' = A(t)x. \tag{2.1}$$

Let also $T(t,\tau)$ be the associated evolution family. Moreover, we consider a family of norms $\|\cdot\|_t$ on X for $t \in \mathbb{R}$ such that:

(i) there exist constants C and $\varepsilon \geq 0$ such that

$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon|t|}\|x\| \tag{2.2}$$

for $x \in X$ and $t \in \mathbb{R}$;

(ii) the map $t \mapsto \|x\|_t$ is measurable for each $x \in X$.

We say that equation (2.1) admits a *strong exponential dichotomy* with respect to the family of norms $\|\cdot\|_t$ if:

(iii) there exist projections $P(t)$ for $t \in \mathbb{R}$ such that

$$P(t)T(t,\tau) = T(t,\tau)P(\tau), \quad t,\tau \in \mathbb{R}; \tag{2.3}$$

(iv) there exist constants

$$\underline{a} \leq \overline{a} < 0 < \underline{b} \leq \overline{b} \quad \text{and} \quad D > 0$$

such that

$$\begin{aligned} \|T(t,\tau)P(\tau)x\|_t &\leq De^{\overline{a}(t-\tau)}\|x\|_\tau, \\ \|T(\tau,t)Q(t)x\|_\tau &\leq De^{-\underline{b}(t-\tau)}\|x\|_t \end{aligned} \tag{2.4}$$

for $t \geq \tau$ and

$$\begin{aligned} \|T(t,\tau)P(\tau)x\|_t &\leq De^{\underline{a}(t-\tau)}\|x\|_\tau, \\ \|T(\tau,t)Q(t)x\|_\tau &\leq De^{-\overline{b}(t-\tau)}\|x\|_t \end{aligned} \tag{2.5}$$

for $t \leq \tau$, where $Q(\tau) = \mathrm{Id} - P(\tau)$.

2.2 From exponential behavior to admissibility

Let Y be the set of all continuous functions $x\colon \mathbb{R} \to X$ such that

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty.$$

One can easily verify that when equipped with the norm $\|\cdot\|_\infty$ the set Y is a Banach space.

We first show that for a strong exponential dichotomy the pair (Y, Y) is admissible in the strong sense, that is, considering classical solutions of equation (2.1).

Theorem 2.1. *Assume that equation (2.1) admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$. Then:*

1. *for each $y \in Y$ there exists a unique $x \in Y$ such that*
$$x'(t) - A(t)x(t) = y(t) \tag{2.6}$$
for $t \in \mathbb{R}$;
2. *there exist $K, a > 0$ such that*
$$\|T(t,\tau)x\|_t \le Ke^{a|t-\tau|}\|x\|_\tau \tag{2.7}$$
for $x \in X$ and $t, \tau \in \mathbb{R}$.

Proof. For the first statement in the theorem, take $y \in Y$. For $t \in \mathbb{R}$ we define

$$x(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)y(\tau)\,d\tau - \int_{t}^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau. \tag{2.8}$$

It follows from (2.4) that

$$\begin{aligned}
&\int_{-\infty}^{t} \|T(t,\tau)P(\tau)y(\tau)\|_t\,d\tau + \int_{t}^{+\infty} \|T(t,\tau)Q(\tau)y(\tau)\|_t\,d\tau \\
&\quad\le D\|y\|_\infty \left(\int_{-\infty}^{t} e^{\bar{a}(t-\tau)} d\tau + \int_{t}^{+\infty} e^{-\underline{b}(\tau-t)} d\tau\right) \\
&\quad= D\left(-\frac{1}{\bar{a}} + \frac{1}{\underline{b}}\right)\|y\|_\infty
\end{aligned} \tag{2.9}$$

for $t \in \mathbb{R}$ and thus, $x(t)$ is well defined. Moreover, given $t_0 \in \mathbb{R}$, we have

$$\begin{aligned}
x(t) &= \int_{t_0}^{t} T(t,\tau)y(\tau)\,d\tau - \int_{t_0}^{t} T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_{t_0}^{t} T(t,\tau)Q(\tau)y(\tau)\,d\tau + \int_{-\infty}^{t} T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_{t}^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau \\
&= \int_{t_0}^{t} T(t,\tau)y(\tau)\,d\tau + \int_{-\infty}^{t_0} T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_{t_0}^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau \\
&= \int_{t_0}^{t} T(t,\tau)y(\tau)\,d\tau + T(t,t_0)x(t_0)
\end{aligned} \tag{2.10}$$

and hence,

$$x(t) = T(t,t_0)x(t_0) + \int_{t_0}^{t} T(t,\tau)y(\tau)\,d\tau \tag{2.11}$$

for $t \in \mathbb{R}$. Since $T(t,\tau)$ is the evolution family of equation (2.1), it follows from (2.11) that the function $x\colon \mathbb{R} \to X$ is differentiable and that identity (2.6) holds for $t \in \mathbb{R}$. Moreover, it follows from (2.9) that $x \in Y$.

Lemma 2.2. *x is the unique function in Y satisfying* (2.6)*.*

Proof of the lemma. Since the map $x \mapsto y$ defined by identity (2.6) is linear, it is sufficient to show that if a function $x \in Y$ satisfies $x'(t) = A(t)x(t)$ for $t \in \mathbb{R}$, then $x = 0$. Let

$$x^s(t) = P(t)x(t) \quad \text{and} \quad x^u(t) = Q(t)x(t).$$

Then $x(t) = x^s(t) + x^u(t)$ and it follows from (2.3) that

$$x^s(t) = T(t,\tau)x^s(\tau) \quad \text{and} \quad x^u(t) = T(t,\tau)x^u(\tau)$$

for $t,\tau \in \mathbb{R}$. Since $x^s(t) = T(t,t-\tau)x^s(t-\tau)$ for $\tau \geq 0$, we have

$$\begin{aligned}
\|x^s(t)\|_t &= \|T(t,t-\tau)x^s(t-\tau)\|_t \\
&= \|T(t,t-\tau)P(t-\tau)x(t-\tau)\|_t \\
&\leq De^{\bar{a}\tau}\|x(t-\tau)\|_{t-\tau} \\
&\leq De^{\bar{a}\tau}\|x\|_\infty
\end{aligned}$$

and letting $\tau \to +\infty$ yields that $x^s(t) = 0$ for $t \in \mathbb{R}$. Similarly, since $x^u(t) = T(t,t+\tau)x^u(t+\tau)$ for $\tau \geq 0$, we have

$$\begin{aligned}
\|x^u(t)\|_t &= \|T(t,t+\tau)x^u(t+\tau)\|_t \\
&= \|T(t,t+\tau)Q(t+\tau)x(t+\tau)\|_t \\
&\leq De^{-\underline{b}\tau}\|x(t+\tau)\|_{t+\tau} \\
&\leq De^{-\underline{b}\tau}\|x\|_\infty
\end{aligned}$$

and hence, $x^u(t) = 0$ for $t \in \mathbb{R}$. Therefore, $x(t) = 0$ for $t \in \mathbb{R}$. ☐

It remains to establish the second statement in the theorem. It follows from (2.4) and (2.5) that

$$\begin{aligned}
\|T(t,\tau)x\|_t &\leq \|T(t,\tau)P(\tau)x\|_t + \|T(t,\tau)Q(\tau)x\|_t \\
&\leq De^{\bar{a}(t-\tau)}\|x\|_\tau + De^{\bar{b}(t-\tau)}\|x\|_\tau \\
&\leq 2De^{\bar{b}(t-\tau)}\|x\|_\tau
\end{aligned}$$

for $t \geq \tau$ and similarly

$$\|T(t,\tau)x\|_t \leq 2De^{-\underline{a}(\tau-t)}\|x\|_\tau$$

for $t \leq \tau$. Therefore, (2.7) holds with $K = 2D$ and $a = \max\{\bar{b}, -\underline{a}\}$. ☐

2.3 From admissibility to exponential behavior

Now we establish the converse of Theorem 2.1, that is, we show that if the pair (Y, Y) is admissible, then equation (2.1) admits a strong exponential dichotomy.

Theorem 2.3. *Assume that for each $y \in Y$ there exists a unique $x \in Y$ such that:*

1. *identity (2.6) holds for $t \in \mathbb{R}$;*
2. *there exist $K, a > 0$ such that (2.7) holds for $x \in X$ and $t, \tau \in \mathbb{R}$.*

Then equation (2.1) admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$.

Proof. Let H be the linear operator defined by

$$(Hx)(t) = x'(t) - A(t)x(t), \quad t \in \mathbb{R} \tag{2.12}$$

in the domain $\mathcal{D}(H)$ formed by all $x \in Y$ such that $Hx \in Y$.

Lemma 2.4. *The operator $H \colon \mathcal{D}(H) \to Y$ is closed.*

Proof of the lemma. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H)$ converging to $x \in Y$ such that $y_k = Hx_k$ converges to $y \in Y$. For each $\tau \in \mathbb{R}$, we have

$$\begin{aligned} x(t) - x(\tau) &= \lim_{k\to\infty} (x_k(t) - x_k(\tau)) \\ &= \lim_{k\to\infty} \int_\tau^t x_k'(s)\, ds \\ &= \lim_{k\to\infty} \int_\tau^t (y_k(s) + A(s)x_k(s))\, ds \end{aligned}$$

for $t \geq \tau$. Moreover, it follows from (2.2) that

$$\begin{aligned} \left\| \int_\tau^t y_k(s)\, ds - \int_\tau^t y(s)\, ds \right\| &\leq \int_\tau^t \|y_k(s) - y(s)\|\, ds \\ &\leq \int_\tau^t \|y_k(s) - y(s)\|_s\, ds \\ &\leq (t-\tau)\|y_k - y\|_\infty. \end{aligned}$$

Since $y_k \to y$ in Y, we obtain

$$\lim_{k\to\infty} \int_\tau^t y_k(s)\, ds = \int_\tau^t y(s)\, ds.$$

Similarly,

$$\begin{aligned} \left\| \int_\tau^t A(s)x_k(s)\, ds - \int_\tau^t A(s)x(s)\, ds \right\| &\leq M \int_\tau^t \|x_k(s) - x(s)\| ds \\ &\leq M(t-\tau)\|x_k - x\|_\infty, \end{aligned}$$

where

$$M = \sup\{\|A(s)\| : s \in [\tau, t]\}.$$

Since the function $s \mapsto A(s)x$ is continuous for each $x \in X$, we have

$$\sup_{\tau \leq s \leq t} \|A(s)x\| < +\infty$$

and it follows from the Banach–Steinhaus theorem that $M < +\infty$. Since $Hx_k \to y$ in Y, we obtain

$$\lim_{k\to\infty} \int_\tau^t A(s)x_k(s)\,ds = \int_\tau^t A(s)x(s)\,ds.$$

Therefore,

$$x(t) - x(\tau) = \int_\tau^t (A(s)x(s) + y(s))\,ds,$$

which implies that $Hx = y$ and $x \in \mathcal{D}(H)$. □

It follows from Lemma 2.4 and the closed graph theorem that the operator H has a bounded inverse $G\colon Y \to Y$.

For $\tau \in \mathbb{R}$, let F_τ^s be the set of all $x \in X$ such that there exists a solution u of equation (2.1) with $u(\tau) = x$ satisfying

$$\sup\left\{\|u(t)\|_t : t \in [\tau, +\infty)\right\} < +\infty. \tag{2.13}$$

Similarly, let F_τ^u be the set of all $x \in X$ such that there exists a solution u of equation (2.1) with $u(\tau) = x$ satisfying

$$\sup\left\{\|u(t)\|_t : t \in (-\infty, \tau]\right\} < +\infty. \tag{2.14}$$

One can easily verify that F_τ^s and F_τ^u are subspaces of X.

Lemma 2.5. *For $\tau \in \mathbb{R}$, we have*

$$X = F_\tau^s \oplus F_\tau^u. \tag{2.15}$$

Proof of the lemma. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, +\infty)$ such that $0 \le \phi \le 1$, $\phi = 1$ on $[\tau + 1, +\infty)$ and $\sup_{t\in\mathbb{R}}|\phi'(t)| < +\infty$. Moreover, given $x \in X$ let u be the solution of equation (2.1) with $u(\tau) = x$. It follows from (2.13) that $g := \phi' u \in Y$. Since H is invertible, there exists $v \in Y$ such that $Hv = g$. Let $w = (1 - \phi)u + v$. One can easily verify that $Hw = 0$. Furthermore,

$$\begin{aligned}\sup\left\{\|w(t)\|_t : t \in [\tau, +\infty)\right\} &\le \sup\left\{\|u(t)\|_t : t \in [\tau, \tau+1]\right\} \\ &\quad + \sup\left\{\|v(t)\|_t : t \in [\tau, +\infty)\right\} < +\infty,\end{aligned}$$

and thus $w(\tau) \in F_\tau^s$. On the other hand, $w - u$ is also a solution of equation (2.1) and

$$\sup\left\{\|(w-u)(t)\|_t : t \in (-\infty, \tau]\right\} = \sup\left\{\|v(t)\|_t : t \in (-\infty, \tau]\right\} < +\infty.$$

Hence,

$$w(\tau) - x = w(\tau) - u(\tau) = v(\tau) \in F_\tau^u$$

and $x \in F_\tau^s + F_\tau^u$.

It remains to show that $F_\tau^s \cap F_\tau^u = \{0\}$. Take $x \in F_\tau^s \cap F_\tau^u$ and let u be the solution of equation (2.1) with $u(\tau) = x$. It follows from (2.13) and (2.14) that $u \in Y$. Since H is invertible, we must have $u = 0$ and hence $x = 0$. □

Now let $P(\tau)\colon X \to F_\tau^s$ and $Q(\tau)\colon X \to F_\tau^u$ be the projections associated to the decomposition in (2.15), with $P(\tau) + Q(\tau) = \mathrm{Id}$. It follows readily from the definitions that property (2.3) holds.

Lemma 2.6. *There exists $M > 0$ such that*

$$\|P(\tau)x\|_\tau \le M\|x\|_\tau \tag{2.16}$$

for $x \in X$ and $\tau \in \mathbb{R}$.

Proof of the lemma. Using the same notation as in the proof of Lemma 2.5, we have

$$\begin{aligned}\|P(\tau)x\|_\tau &= \|w(\tau)\|_\tau \le \|u(\tau)\|_\tau + \|v(\tau)\|_\tau \\ &\le \|x\|_\tau + \|v\|_\infty = \|x\|_\tau + \|Gg\|_\infty.\end{aligned} \tag{2.17}$$

Furthermore,

$$\|g\|_\infty = \|\phi' u\|_\infty \le L \sup\{\|u(t)\|_t : t \in [\tau, \tau+1]\},$$

where $L = \sup_{t\in\mathbb{R}} |\phi'(t)|$. We note that the constant L is independent of τ. Using (2.7) we obtain

$$\|g\|_\infty \le LKe^a \|x\|_\tau$$

and it follows from (2.17) that

$$\|P(\tau)x\|_\tau \le (1 + \|G\| LKe^a)\|x\|_\tau.$$

This shows that (2.16) holds taking $M = 1 + \|G\| LKe^a$. □

Lemma 2.7. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)P(\tau)x\|_t \le De^{-\lambda(t-\tau)}\|x\|_\tau \tag{2.18}$$

for $x \in X$ and $t \ge \tau$.

Proof of the lemma. Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, +\infty)$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[\tau+1, +\infty)$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Moreover, given $x \in F^s_\tau$, let u be the solution of equation (2.1) with $u(\tau) = x$. It follows from (2.13) that $\psi u \in Y$ and one can easily verify that $H(\psi u) = \psi' u$. Moreover,

$$\begin{aligned}\sup\{\|u(t)\|_t : t \in [\tau+1, +\infty)\} &= \sup\{\|\psi(t)u(t)\|_t : t \in [\tau+1, +\infty)\} \\ &\le \|\psi u\|_\infty = \|G(\psi' u)\|_\infty \\ &\le \|G\| \cdot \|\psi' u\|_\infty \\ &= \|G\| \sup\{\|(\psi' u)(t)\|_t : t \in [\tau, \tau+1]\} \\ &\le 2\|G\| \sup\{\|u(t)\|_t : t \in [\tau, \tau+1]\} \\ &= 2\|G\| \sup\{\|T(t,\tau)u(\tau)\|_t : t \in [\tau, \tau+1]\} \\ &\le 2Ke^a\|G\| \cdot \|u(\tau)\|_\tau \\ &= 2Ke^a\|G\| \cdot \|x\|_\tau,\end{aligned}$$

using (2.7) in the last inequality. Hence, using again (2.7), we obtain

$$\|u(t)\|_t \le C\|x\|_\tau \quad \text{for} \quad t \ge \tau, \tag{2.19}$$

where $C = 2Ke^a \max\{1, \|G\|\}$.

Now we show that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F^s_\tau$,

$$\|u(t)\|_t \le \frac{1}{2}\|x\|_\tau \quad \text{for} \quad t - \tau \ge N. \tag{2.20}$$

In order to prove (2.20), take $t_0 \in \mathbb{R}$ such that $t_0 > \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (2.19) that

$$\frac{1}{2C}\|x\|_\tau < \|u(s)\|_s \le C\|x\|_\tau, \quad \tau \le s \le t_0. \tag{2.21}$$

Now take $\varepsilon > 0$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, t_0]$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $[\tau+\varepsilon, t_0-\varepsilon]$. Moreover, let

$$y(t) = \psi(t)u(t) \quad \text{and} \quad v(t) = u(t)\int_{-\infty}^{t} \psi(s)\,ds$$

for $t \in \mathbb{R}$. Clearly, y and v belong to Y and one can easily verify that $Hv = y$. Therefore,

$$\|G\| \sup\{\|u(t)\|_t : t \in [\tau, t_0]\} \ge \|G\| \cdot \|y\|_\infty \ge \|v\|_\infty.$$

Hence, it follows from (2.21) that

$$\begin{aligned} C\|G\| \cdot \|x\|_\tau &\ge \|v(t_0)\|_{t_0} \\ &\ge \|u(t_0)\|_{t_0} \int_{\tau+\varepsilon}^{t_0-\varepsilon} \psi(s)\,ds \\ &\ge \frac{1}{2C}(t_0 - \tau - 2\varepsilon)\|x\|_\tau. \end{aligned}$$

Letting $\varepsilon \to 0$ yields the inequality

$$t_0 - \tau \le 2C^2\|G\|.$$

Hence, property (2.20) holds taking $N > 2C^2\|G\|$.

In order to complete the proof, take $t \ge \tau$ and write $t - \tau = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (2.16), (2.19) and (2.20), we obtain

$$\begin{aligned} \|T(t,\tau)P(\tau)x\|_t &= \|T(\tau+kN+r,\tau)P(\tau)x\|_{\tau+kN+r} \\ &\le \frac{1}{2^k}\|T(\tau+r,\tau)P(\tau)x\|_{\tau+r} \\ &\le \frac{C}{2^k}\|P(\tau)x\|_\tau \\ &\le 2CMe^{-(t-\tau)\log 2/N}\|x\|_\tau, \end{aligned}$$

for $x \in X$. Taking $D = 2CM$ and $\lambda = \log 2/K$ yields inequality (2.18). □

Lemma 2.8. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)Q(\tau)x\|_t \le De^{-\lambda(\tau-t)}\|x\|_\tau \tag{2.22}$$

for $x \in X$ and $t \le \tau$.

Proof of the lemma. Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\infty, \tau]$ such that $0 \le \psi \le 1$, $\psi = 1$ on $(-\infty, \tau-1]$ and $\sup_{t\in\mathbb{R}}|\psi'(t)| \le 2$. Moreover, given $x \in F_\tau^u$, let u be the solution of equation (2.1) with $u(\tau) = x$. It follows from (2.14) that $\psi u \in Y$ and one can easily verify that $H(\psi u) = \psi' u$. Moreover,

$$\begin{aligned} \sup\{\|u(t)\|_t : t \in (-\infty, \tau-1]\} &= \sup\{\|\psi(t)u(t)\|_t : t \in (-\infty, \tau-1]\} \\ &\le \|\psi u\|_\infty = \|G(\psi' u)\|_\infty \\ &\le \|G\| \cdot \|\psi' u\|_\infty \\ &= \|G\| \sup\{\|(\psi' u)(t)\|_t : t \in [\tau-1, \tau]\} \\ &\le 2\|G\| \sup\{\|u(t)\|_t : t \in [\tau-1, \tau]\} \\ &= 2\|G\| \sup\{\|T(t,\tau)u(\tau)\|_t : t \in [\tau-1, \tau]\} \\ &\le 2Ke^a\|G\| \cdot \|u(\tau)\|_\tau = 2Ke^a\|G\| \cdot \|x\|_\tau, \end{aligned}$$

using (2.7) in the last inequality. Hence, using again (2.7), we obtain

$$\|u(t)\|_t \le C\|x\|_\tau \quad \text{for} \quad t \le \tau, \tag{2.23}$$

where $C = 2Ke^a \max\{1, \|G\|\}$.

We also show that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F_\tau^u$,

$$\|u(t)\|_t \le \frac{1}{2}\|x\|_\tau \quad \text{for} \quad \tau - t \ge N. \tag{2.24}$$

In order to prove (2.24), take $t_0 \in \mathbb{R}$ such that $t_0 < \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (2.23) that

$$\frac{1}{2C}\|x\|_\tau < \|u(s)\|_s \le C\|x\|_\tau, \quad t_0 \le s \le \tau. \tag{2.25}$$

Now take $\varepsilon > 0$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[t_0, \tau]$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $[t_0 + \varepsilon, \tau - \varepsilon]$. Moreover, let

$$y(t) = -\psi(t)u(t) \quad \text{and} \quad v(t) = u(t)\int_t^{+\infty} \psi(s)\, ds$$

for $t \in \mathbb{R}$. Clearly, y and v belong to Y and one can easily verify that $Hv = y$. Therefore,

$$\|G\| \sup\left\{\|u(t)\|_t : t \in [t_0, \tau]\right\} \ge \|G\| \cdot \|y\|_\infty \ge \|v\|_\infty.$$

Hence, it follows from (2.25) that

$$\begin{aligned} C\|G\| \cdot \|x\|_\tau &\ge \|v(t_0)\|_{t_0} \\ &\ge \|u(t_0)\|_{t_0} \int_{t_0+\varepsilon}^{\tau-\varepsilon} \psi(s)\, ds \\ &\ge \frac{1}{2C}(\tau - t_0 - 2\varepsilon)\|x\|_\tau. \end{aligned}$$

Letting $\varepsilon \to 0$ yields the inequality

$$\tau - t_0 \le 2C^2\|G\|.$$

Hence, property (2.24) holds taking $N > 2C^2\|G\|$.

Finally, take $t \le \tau$ and write $\tau - t = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (2.16), (2.23) and (2.24), we obtain

$$\begin{aligned} \|T(t,\tau)Q(\tau)x\|_t &= \|T(\tau - kN - r, \tau)Q(\tau)x\|_{\tau-kN-r} \\ &\le \frac{1}{2^k}\|T(\tau - r, \tau)Q(\tau)x\|_{\tau-r} \\ &\le \frac{C}{2^k}\|Q(\tau)x\|_\tau \\ &\le 2C(1+M)e^{-(\tau-t)\log 2/N}\|x\|_\tau, \end{aligned}$$

for $x \in X$. Taking $D = 2C(1+M)$ and $\lambda = \log 2/K$ yields inequality (2.22). $\square$

In order to complete the proof of the theorem, we note that it follows from (2.18) and (2.22) that (2.4) holds taking $\overline{a} = -\lambda$ and $\underline{b} = \lambda$. Moreover, it follows from (2.7) and (2.16) that (2.5) holds taking $D = K(1+M)$, $\underline{a} = -a$ and $\overline{b} = a$. $\square$

3 Strong robustness

In this section we establish the robustness of the notion of a strong exponential dichotomy using its characterization in terms of admissibility of the pair (Y, Y) in Theorems 2.1 and 2.3.

Theorem 3.1. *Let $A, B\colon \mathbb{R} \to B(X)$ be strongly continuous functions such that:*

1. *equation (2.1) admits a strong exponential dichotomy with respect to a family of norms $\|\cdot\|_t$ satisfying (2.2) for some $C > 0$ and $\varepsilon \geq 0$;*
2. *there exists $c \geq 0$ such that*

$$\|B(t) - A(t)\| \leq ce^{-\varepsilon|t|}, \quad t \in \mathbb{R}. \tag{3.1}$$

If c is sufficiently small, then the equation $x' = B(t)x$ admits a strong exponential dichotomy with respect to the same family of norms.

Proof. Let H be the linear operator defined by (2.12) on the domain $\mathcal{D}(H)$. For $x \in \mathcal{D}(H)$ we consider the graph norm

$$\|x\|'_\infty = \|x\|_\infty + \|Hx\|_\infty.$$

Clearly, the operator

$$H\colon (\mathcal{D}(H), \|\cdot\|'_\infty) \to (Y, \|\cdot\|_\infty)$$

is bounded. For simplicity, we denote it from now on simply by H. It follows from Lemma 2.4 that $(\mathcal{D}(H), \|\cdot\|'_\infty)$ is a Banach space.

It follows from (2.7) and (3.1) that

$$\|(B(t) - A(t))x\|_t \leq cC\|x\|_t \tag{3.2}$$

for $x \in X$ and $t \in \mathbb{R}$. We define a linear operator $L\colon \mathcal{D}(H) \to Y$ by

$$(Lx)(t) = x'(t) - B(t)x(t), \quad t \in \mathbb{R}.$$

By (3.2) we have

$$\|(H - L)x\|_\infty \leq cC\|x\|'_\infty \tag{3.3}$$

for $x \in \mathcal{D}(T)$. By Theorem 2.1, the operator H is invertible. Hence, it follows from (3.3) that if c is sufficiently small, then L is also invertible. Furthermore, it follows from Theorem 2.1 that there exist constants $K, a > 0$ such that (2.7) holds for $x \in X$ and $t, \tau \in \mathbb{R}$. Now let $U(t, \tau)$ be the evolution family associated to the linear equation $x' = B(t)x$.

Lemma 3.2. *There exist constants $K', a' > 0$ such that*

$$\|U(t, \tau)x\|_t \leq K'e^{a'|t-\tau|}\|x\|_\tau$$

for $x \in X$ and $t, \tau \in \mathbb{R}$.

Proof of the lemma. Let $x(t)$ be a solution of the equation $x' = B(t)x$. For each $t \geq \tau$ we have

$$\begin{aligned}
\|x(t)\|_t &= \left\| T(t, \tau)x(\tau) + \int_\tau^t T(t, s)(B(s) - A(s))x(s)\, ds \right\|_t \\
&\leq Ke^{a(t-\tau)}\|x(\tau)\|_\tau + K\int_\tau^t e^{a(t-s)}\|(B(s) - A(s))x(s)\|_s\, ds \\
&\leq Ke^{a(t-\tau)}\|x(\tau)\|_\tau + cCK\int_\tau^t e^{a(t-s)}\|x(s)\|_s\, ds.
\end{aligned}$$

This shows that the function $\phi(t) = e^{-at}\|x(t)\|_t$ satisfies

$$\phi(t) \le K\phi(\tau) + cCK \int_\tau^t \phi(s)\, ds$$

and using Gronwall's lemma we obtain

$$\phi(t) \le K\phi(\tau)e^{cCK(t-\tau)}.$$

Hence,

$$\|U(t,\tau)x\|_t \le Ke^{(a+cCK)(t-\tau)}\|x\|_\tau$$

for $t \ge \tau$. One can argue in a similar manner for $t \le \tau$. □

Since L is invertible, it follows from Theorem 2.3 together with Lemma 3.2 that the equation $x' = B(t)x$ admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$. □

4 Exponential behavior and weak admissibility

In this section we consider a weak form of the admissibility property and we use it to give a characterization of the notion of a strong exponential dichotomy.

A family $T(t,\tau)$, for $t,\tau \in \mathbb{R}$, of bounded linear operators on X is said to be an *evolution family* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in \mathbb{R}$;
2. $T(t,s)T(s,\tau) = T(t,\tau)$ for $t,s,\tau \in \mathbb{R}$;
3. given $t,\tau \in \mathbb{R}$ and $x \in X$, the maps $s \mapsto T(t,s)x$ and $s \mapsto T(s,\tau)x$ are continuous.

We continue to consider a family of norms $\|\cdot\|_t$ satisfying conditions (i) and (ii). We say that an evolution family $T(t,s)$ admits a *strong exponential dichotomy* with respect to the family of norms $\|\cdot\|_t$ if conditions (iii) and (iv) hold.

We first show that the existence of a strong exponential dichotomy yields the weak admissibility of the pair (Y,Y).

Theorem 4.1. *If the evolution family $T(t,\tau)$ admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$, then:*

1. *for each $y \in Y$ there exists a unique $x \in Y$ such that*
$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\, ds \quad \text{for} \quad t \ge \tau; \tag{4.1}$$
2. *there exist $K, a > 0$ such that (2.7) holds.*

Proof. Take $y \in Y$. For $t \in \mathbb{R}$ we define $x(t)$ as in (2.8). Then (2.9) holds and proceeding as in (2.10) we obtain

$$x(t) = \int_\tau^t T(t,s)y(s)\, ds + T(t,\tau)x(\tau)$$

for $t \ge \tau$. This shows that property (4.1) holds. It follows readily from (4.1) that the function x is continuous and thus $x \in Y$. The uniqueness of x follows from Lemma 2.2 (that can be obtained using the same proof). This establishes the first property of the theorem.

The second property follows exactly as in the proof of Theorem 2.1. □

Now we establish the converse of Theorem 4.1.

Theorem 4.2. *Assume that for each $y \in Y$ there exists a unique $x \in Y$ such that (4.1) holds and that there exist constants $K, a > 0$ such that (2.7) holds for $x \in X$ and $t, \tau \in \mathbb{R}$. Then the evolution family $T(t,\tau)$ admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$.*

Proof. Let H be the linear operator defined by $Hx = y$ in the domain $\mathcal{D}(H)$ formed by all $x \in Y$ for which there exists $y \in Y$ satisfying (4.1). In order to show that H is well defined, let $y_1, y_2 \in Y$ be such that

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y_1(s)\,ds$$

and

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y_2(s)\,ds$$

for $t \geq \tau$. Then

$$\frac{1}{t-\tau}\int_\tau^t T(t,s)y_1(s)\,ds = \frac{1}{t-\tau}\int_\tau^t T(t,s)y_2(s)\,ds$$

and since the map $s \mapsto T(t,s)y_i(s)$ is continuous for $i = 1, 2$, letting $\tau \to t$ yields that $y_1(t) = y_2(t)$ for $t \in \mathbb{R}$.

Lemma 4.3. *The operator $H\colon \mathcal{D}(H) \to Y$ is closed.*

Proof of the lemma. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H)$ converging to $x \in Y$ such that Hx_n converges to $y \in Y$. For each $\tau \in \mathbb{R}$, we have

$$\begin{aligned} x(t) - T(t,\tau)x(\tau) &= \lim_{n\to\infty}(x_n(t) - T(t,\tau)x_n(\tau)) \\ &= \lim_{n\to\infty}\int_\tau^t T(t,s)y_n(s)\,ds \end{aligned}$$

for $t \geq \tau$. Furthermore,

$$\begin{aligned} \left\| \int_\tau^t T(t,s)y_n(s)\,ds - \int_\tau^t T(t,s)y(s)\,ds \right\| &\leq M\int_\tau^t \|y_n(s) - y(s)\|\,ds \\ &\leq M\int_\tau^t \|y_n(s) - y(s)\|_s\,ds \\ &\leq M\|y_n - y\|_\infty(t-\tau), \end{aligned}$$

where

$$M = \sup\big\{\|T(t,s)\| : s \in [\tau,t]\big\}.$$

Since the map $s \mapsto T(t,s)x$ is continuous for each $x \in X$, we have

$$\sup_{\tau\leq s\leq t}\|T(t,s)x\| < +\infty$$

and it follows from the Banach–Steinhaus theorem that $M < +\infty$. Since y_n converges to y in Y, we conclude that

$$\lim_{n\to\infty}\int_\tau^t T(t,s)y_n(s)\,ds = \int_\tau^t T(t,s)y(s)\,ds$$

and

$$x(t) - T(t,\tau)x(\tau) = \int_\tau^t T(t,s)y(s)\,ds$$

for $t \geq \tau$. This shows that (4.1) holds. Hence, $Hx = y$ and $x \in \mathcal{D}(H)$. □

It follows from the closed graph theorem that H has a bounded inverse $G\colon Y \to Y$. For each $\tau \in \mathbb{R}$ we define

$$F_\tau^s = \left\{ x \in X : \sup_{t \ge \tau} \|T(t,\tau)x\|_t < +\infty \right\}$$

and

$$F_\tau^u = \left\{ x \in X : \sup_{t \le \tau} \|T(t,\tau)x\|_t < +\infty \right\}.$$

One can easily verify that F_τ^s and F_τ^u are subspaces of X.

Lemma 4.4. *For $\tau \in \mathbb{R}$, we have*

$$X = F_\tau^s \oplus F_\tau^u. \tag{4.2}$$

Proof of the lemma. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[\tau, \tau+1]$ such that $\int_\tau^{\tau+1} \phi(s)\, ds = 1$. Given $x \in X$, we define a function $g\colon \mathbb{R} \to X$ by

$$g(t) = \phi(t) T(t,\tau)x.$$

Clearly, $g \in Y$. Since H is invertible, there exists $v \in Y$ such that $Hv = g$. Moreover, it follows from (4.1) that

$$v(t) = T(t,\tau)(v(\tau) + x)$$

for $t \ge \tau + 1$ and thus $v(\tau) + x \in F_\tau^s$. Furthermore, again by (4.1), we have $v(t) = T(t,\tau)v(\tau)$ for $t \le \tau$ and thus $v(\tau) \in F_\tau^u$. This shows that $x \in F_\tau^s + F_\tau^u$.

Now take $x \in F_\tau^s \cap F_\tau^u$. We define a function $u\colon \mathbb{R} \to X$ by $u(t) = T(t,\tau)x$. It follows from the definitions of F_τ^s and F_τ^u that $u \in Y$. Moreover, $Hu = 0$ and $u \in \mathcal{D}(H)$. Since H is invertible, we obtain $u = 0$ and hence $x = 0$. $\square$

Now let $P(\tau)\colon X \to F_\tau^s$ and $Q(\tau)\colon X \to F_\tau^u$ be the projections associated to the decomposition in (4.2), with $P(\tau) + Q(\tau) = \mathrm{Id}$.

Lemma 4.5. *There exists $M > 0$ such that*

$$\|P(\tau)x\|_\tau \le M\|x\|_\tau \tag{4.3}$$

for $x \in X$ and $\tau \in \mathbb{R}$.

Proof of the lemma. Using the same notation as in Lemma 4.4, we have

$$\begin{aligned} \|P(\tau)x\|_\tau &= \|v(\tau) + x\|_\tau \\ &\le \|v(\tau)\|_\tau + \|x\|_\tau \le \|v\|_\infty + \|x\|_\tau \\ &= \|Gg\|_\infty + \|x\|_\tau \le \|G\| \cdot \|g\|_\infty + \|x\|_\tau. \end{aligned}$$

On the other hand, it follows from (2.7) that $\|g\|_\infty \le CKe^a\|x\|_\tau$, where

$$C = \sup\left\{ |\phi(t)| : t \in [\tau, \tau+1] \right\}.$$

This shows that (4.3) holds taking $M = CKe^a\|G\| + 1$. $\square$

Lemma 4.6. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)P(\tau)x\|_t \le De^{-\lambda(t-\tau)}\|x\|_\tau \tag{4.4}$$

for $x \in X$ and $t \ge \tau$.

Proof of the lemma. Take $x \in F^s_\tau$ and let $u(t) = T(t,\tau)x$. Moreover, let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, +\infty)$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[\tau+1, +\infty)$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Clearly, $\psi u \in Y$ and one can easily verify that $H(\psi u) = \psi' u$. Moreover,

$$\begin{aligned}
\sup\{\|u(t)\|_t : t \in [\tau+1,+\infty)\} &= \sup\{\|\psi(t)u(t)\|_t : t \in [\tau+1,+\infty)\}\\
&\le \|\psi u\|_\infty = \|G(\psi' u)\|_\infty\\
&\le \|G\| \cdot \|\psi' u\|_\infty\\
&= \|G\| \sup\{\|(\psi' u)(t)\|_t : t \in [\tau, \tau+1]\}\\
&\le 2\|G\| \sup\{\|u(t)\|_t : t \in [\tau,\tau+1]\}\\
&= 2\|G\| \sup\{\|T(t,\tau)u(\tau)\|_t : t \in [\tau,\tau+1]\}\\
&\le 2Ke^a\|G\| \cdot \|u(\tau)\|_\tau = 2Ke^a\|G\| \cdot \|x\|_\tau,
\end{aligned}$$

using (2.7) in the last inequality. Hence, again using (2.7), we obtain

$$\|u(t)\|_t \le C\|x\|_\tau \quad \text{for} \quad t \ge \tau, \tag{4.5}$$

where $C = 2Ke^a \max\{1, \|G\|\}$.

We show that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F^s_\tau$,

$$\|u(t)\|_t \le \frac{1}{2}\|x\|_\tau \quad \text{for} \quad t - \tau \ge N. \tag{4.6}$$

In order to prove (4.4), take $t_0 \in \mathbb{R}$ such that $t_0 > \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (4.5) that

$$\frac{1}{2C}\|x\|_\tau < \|u(s)\|_s \le C\|x\|_\tau, \quad \tau \le s \le t_0. \tag{4.7}$$

Now take $\varepsilon > 0$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, t_0]$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $[\tau+\varepsilon, t_0-\varepsilon]$. Moreover, let

$$y(t) = \psi(t)u(t) \quad \text{and} \quad v(t) = u(t)\int_{-\infty}^t \psi(s)\,ds$$

for $t \in \mathbb{R}$. Clearly, y and v belong to Y and one can easily verify that $Hv = y$. Therefore,

$$\|G\| \sup\{\|u(t)\|_t : t \in [\tau, t_0]\} \ge \|G\| \cdot \|y\|_\infty \ge \|v\|_\infty.$$

Hence, it follows from (4.7) that

$$\begin{aligned}
C\|G\| \cdot \|x\|_\tau &\ge \|v(t_0)\|_{t_0}\\
&\ge \|u(t_0)\|_{t_0} \int_{\tau+\varepsilon}^{t_0-\varepsilon} \psi(s)\,ds\\
&\ge \frac{1}{2C}(t_0 - \tau - 2\varepsilon)\|x\|_\tau.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields the inequality

$$t_0 - \tau \le 2C^2\|G\|.$$

Hence, property (4.6) holds taking $N > 2C^2\|G\|$.

Now take $t \geq \tau$ and write $t-\tau = kN + r$, with $k \in \mathbb{N}$ and $0 \leq r < N$. By (4.3), (4.5) and (4.6), we obtain

$$\begin{aligned}\|T(t,\tau)P(\tau)x\|_t &= \|T(\tau+kN+r,\tau)P(\tau)x\|_{\tau+kN+r} \\ &\leq \frac{1}{2^k}\|T(\tau+r,\tau)P(\tau)x\|_{\tau+r} \\ &\leq \frac{C}{2^k}\|P(\tau)x\|_\tau \\ &\leq 2CMe^{-(t-\tau)\log 2/N}\|x\|_\tau,\end{aligned}$$

for $x \in X$. Taking $D = 2CM$ and $\lambda = \log 2/K$ yields property (4.4). □

Lemma 4.7. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)Q(\tau)x\|_t \leq De^{-\lambda(\tau-t)}\|x\|_\tau \tag{4.8}$$

for $x \in X$ and $t \leq \tau$.

Proof of the lemma. Take $x \in F_\tau^u$ and let $u(t) = T(t,\tau)x$. Moreover, let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\infty,\tau]$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on $(-\infty,\tau-1]$ and $\sup_{t\in\mathbb{R}}|\psi'(t)| \leq 2$. Clearly, $\psi u \in Y$ and one can easily verify that $H(\psi u) = \psi' u$. Moreover,

$$\begin{aligned}\sup\left\{\|u(t)\|_t : t \in (-\infty,\tau-1]\right\} &= \sup\left\{\|\psi(t)u(t)\|_t : t \in (-\infty,\tau-1]\right\} \\ &\leq \|\psi u\|_\infty = \|G(\psi' u)\|_\infty \\ &\leq \|G\|\cdot\|\psi' u\|_\infty \\ &= \|G\|\sup\left\{\|(\psi' u)(t)\|_t : t \in [\tau-1,\tau]\right\} \\ &\leq 2\|G\|\sup\left\{\|u(t)\|_t : t \in [\tau-1,\tau]\right\} \\ &= 2\|G\|\sup\left\{\|T(t,\tau)u(\tau)\|_t : t \in [\tau-1,\tau]\right\} \\ &\leq 2Ke^a\|G\|\cdot\|u(\tau)\|_\tau \\ &= 2Ke^a\|G\|\cdot\|x\|_\tau,\end{aligned}$$

using (2.7) in the last inequality. Hence, again using (2.7), we obtain

$$\|u(t)\|_t \leq C\|x\|_\tau \quad \text{for} \quad t \leq \tau, \tag{4.9}$$

where $C = 2Ke^a\max\{1,\|G\|\}$.

Now we show that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F_\tau^u$,

$$\|u(t)\|_t \leq \frac{1}{2}\|x\|_\tau \quad \text{for} \quad \tau - t \geq N. \tag{4.10}$$

In order to prove (4.10), take $t_0 \in \mathbb{R}$ such that $t_0 < \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (4.9) that

$$\frac{1}{2C}\|x\|_\tau < \|u(s)\|_s \leq C\|x\|_\tau, \quad t_0 \leq s \leq \tau. \tag{4.11}$$

Now take $\varepsilon > 0$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[t_0,\tau]$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $[t_0+\varepsilon,\tau-\varepsilon]$. Moreover, let

$$y(t) = -\psi(t)u(t) \quad \text{and} \quad v(t) = u(t)\int_t^{+\infty}\psi(s)\,ds$$

for $t \in \mathbb{R}$. Clearly, y and v belong to Y and one can easily verify that $Hv = y$. Therefore,

$$\|G\| \sup \{\|u(t)\|_t : t \in [t_0, \tau]\} \geq \|G\| \cdot \|y\|_\infty \geq \|v\|_\infty.$$

Hence, it follows from (4.11) that

$$\begin{aligned} C\|G\| \cdot \|x\|_\tau &\geq \|v(t_0)\|_{t_0} \\ &\geq \|u(t_0)\|_{t_0} \int_{t_0+\varepsilon}^{\tau-\varepsilon} \psi(s)\, ds \\ &\geq \frac{1}{2C}(\tau - t_0 - 2\varepsilon)\|x\|_\tau. \end{aligned}$$

Letting $\varepsilon \to 0$ yields the inequality

$$\tau - t_0 \leq 2C^2\|G\|.$$

Hence, property (4.8) holds taking $N > 2C^2\|G\|$.

Finally, take $t \leq \tau$ and write $\tau - t = kN + r$, with $k \in \mathbb{N}$ and $0 \leq r < N$. By (4.3), (4.9) and (4.10), we obtain

$$\begin{aligned} \|T(t,\tau)Q(\tau)x\|_t &= \|T(\tau - kN - r, \tau)Q(\tau)x\|_{\tau-kN-r} \\ &\leq \frac{1}{2^k}\|T(\tau - r, \tau)Q(\tau)x\|_{\tau-r} \\ &\leq \frac{C}{2^k}\|Q(\tau)x\|_\tau \\ &\leq 2C(1+M)e^{-(\tau-t)\log 2/N}\|x\|_\tau, \end{aligned}$$

for $x \in X$. Taking $D = 2C(1+M)$ and $\lambda = \log 2/K$ yields property (4.8). □

If follows from (4.4) and (4.8) that (2.4) holds with $\overline{a} = -\lambda$ and $\underline{b} = \lambda$. Moreover, it follows from (2.7) and (4.3) that (2.5) holds with $D = (1+M)K$, $\underline{a} = -a$ and $\overline{b} = a$. This completes the proof of the theorem. □

5 Weak robustness

In a similar manner to that in Section 3 we establish, once more, the robustness of the notion of a strong exponential dichotomy but now using its characterization in terms of the weak admissibility of the pair (Y, Y) in Theorems 4.1 and 4.2.

Theorem 5.1. *Assume that the evolution family $T(t,\tau)$ admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$ and that $B\colon \mathbb{R} \to B(X)$ is a strongly continuous function such that*

$$\|B(t)\| \leq ce^{-\varepsilon|t|}, \quad t \in \mathbb{R}. \tag{5.1}$$

If c is sufficiently small, then the evolution family $U(t,\tau)$ defined by

$$U(t,\tau)x = T(t,\tau)x + \int_\tau^t T(t,s)B(s)U(s,\tau)x\, ds, \quad t,\tau \in \mathbb{R}$$

admits a strong exponential dichotomy with respect to the same family of norms.

Proof. Let L be the linear operator associated to the evolution family $U(t,\tau)$, defined by $Lx = y$ on the domain $\mathcal{D}(L)$ formed by all $x \in Y$ for which there exists $y \in Y$ such that

$$x(t) = U(t,\tau)x(\tau) + \int_\tau^t U(t,s)y(s)\,ds \quad \text{for} \quad t \geq \tau.$$

For each $x, y \in Y$ such that $Lx = y$, we have

$$\begin{aligned}
x(t) &= U(t,\tau)x(\tau) + \int_\tau^t U(t,s)y(s)\,ds \\
&= T(t,\tau)x(\tau) + \int_\tau^t T(t,s)B(s)U(s,\tau)x(\tau)\,ds \\
&\quad + \int_\tau^t T(t,s)y(s)\,ds + \int_\tau^t \int_s^t T(t,w)B(w)U(w,s)y(s)\,dw\,ds \\
&= T(t,\tau)x(\tau) + \int_\tau^t T(t,w)B(w)U(w,\tau)x(\tau)\,dw \\
&\quad + \int_\tau^t T(t,s)y(s)\,ds + \int_\tau^t \int_\tau^w T(t,w)B(w)U(w,s)y(s)\,ds\,dw \\
&= T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \\
&\quad + \int_\tau^t T(t,w)B(w)\left(U(w,\tau)x(\tau) + \int_\tau^w U(w,s)y(s)\,ds\right)dw \\
&= T(t,\tau)x(\tau) + \int_\tau^t T(t,w)(y(w) + B(w)x(w))\,dw
\end{aligned} \tag{5.2}$$

for $t \geq \tau$. Now we introduce an operator $P \colon Y \to Y$ by $(Px)(t) = B(t)x(t)$. It follows from (2.2) and (5.1) that

$$\begin{aligned}
\|B(t)x(t)\|_t &\leq Ce^{\varepsilon|t|}\|B(t)x(t)\| \\
&\leq cC\|x(t)\| \leq cC\|x(t)\|_t
\end{aligned} \tag{5.3}$$

for $t \in \mathbb{R}$ and hence, P is a well defined bounded linear operator. Furthermore, it follows from (5.2) that $\mathcal{D}(H) = \mathcal{D}(L)$ and that $H = L + P$. For $x \in \mathcal{D}(H)$ we consider the graph norm

$$\|x\|'_\infty = \|x\|_\infty + \|Hx\|_\infty.$$

Clearly, the operator

$$H \colon (\mathcal{D}(H), \|\cdot\|'_\infty) \to (Y, \|\cdot\|_\infty)$$

is bounded and for simplicity we denote it simply by H. Moreover, since H is closed, $(\mathcal{D}(H), \|\cdot\|'_\infty)$ is a Banach space. By (5.3) we have

$$\|(H - L)x\|_\infty \leq cC\|x\|_\infty \leq cC\|x\|'_\infty \tag{5.4}$$

for $x \in X$. On the other hand, by Theorem 4.1, the operator H is invertible. Hence, it follows from (5.4) that if c is sufficiently small, then L is also invertible.

It remains to show that there exist $K', a' > 0$ such that

$$\|U(t,\tau)x\|_t \leq K'e^{a'|t-\tau|}\|x\|_\tau \quad \text{for} \quad t, \tau \in \mathbb{R}. \tag{5.5}$$

By (2.7), we have

$$\begin{aligned}\|U(t,\tau)x\|_t &= \left\|T(t,\tau)x + \int_\tau^t T(t,s)B(s)U(s,\tau)x\,ds\right\|_t \\ &\le Ke^{a(t-\tau)}\|x\|_\tau + K\int_\tau^t e^{a(t-s)}\|B(s)U(s,\tau)x\|_s\,ds \\ &\le Ke^{a(t-\tau)}\|x\|_\tau + cCK\int_\tau^t e^{a(t-s)}\|U(s,\tau)x\|_s\,ds\end{aligned}$$

for $t \ge \tau$. Hence, the function $\phi(t) = e^{-at}\|U(t,\tau)x\|_t$ satisfies

$$\phi(t) \le K\phi(\tau) + cCK\int_\tau^t \phi(s)\,ds$$

and it follows from Gronwall's lemma that

$$\phi(t) \le K\phi(\tau)e^{cCK(t-\tau)}.$$

This shows that property (5.5) holds for $t \ge \tau$ taking $K' = K\phi(\tau)$ and $a' = cCK$. A similar argument can be used for $t \le \tau$. One can now apply Theorem 4.2 to conclude that the evolution family $U(t,\tau)$ admits a strong exponential dichotomy. □

6 Strong nonuniform exponential dichotomies

In this section we consider briefly the notion of a strong *nonuniform* exponential dichotomy and we obtain a corresponding robustness result.

We say that an evolution family $T(t,\tau)$, for $t,\tau \in \mathbb{R}$, admits a *strong nonuniform exponential dichotomy* if there exists:

(i) projections $P(t)$ for $t \in \mathbb{R}$ satisfying (2.3);

(ii) constants

$$\underline{\lambda} \le \overline{\lambda} < 0 < \underline{\mu} \le \overline{\mu}, \quad \varepsilon \ge 0 \quad \text{and} \quad D > 0$$

such that

$$\begin{aligned}\|T(t,\tau)P(\tau)x\| &\le De^{\overline{\lambda}(t-\tau)+\varepsilon|\tau|}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\le De^{-\underline{\mu}(t-\tau)+\varepsilon|t|}\|x\|\end{aligned}$$

for $t \ge \tau$ and

$$\begin{aligned}\|T(t,\tau)P(\tau)x\| &\le De^{\underline{\lambda}(t-\tau)+\varepsilon|\tau|}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\le De^{-\overline{\mu}(t-\tau)+\varepsilon|t|}\|x\|\end{aligned}$$

for $t \le \tau$, where $Q(\tau) = \mathrm{Id} - P(\tau)$.

We first relate this notion to the notion of a strong exponential dichotomy with respect to a family of norms.

Proposition 6.1. *The following properties are equivalent:*

1. *$T(t,\tau)$ admits a strong nonuniform exponential dichotomy;*
2. *$T(t,\tau)$ admits a strong exponential dichotomy with respect to a family of norms $\|\cdot\|_t$ satisfying conditions (i) and (ii).*

Proof. Assume that $T(t,\tau)$ admits a strong nonuniform exponential dichotomy. For $x \in X$ and $\tau \in \mathbb{R}$, write $y = P(\tau)x$ and $z = Q(\tau)x$, and let

$$\|x\|_\tau = \max\left\{\|y\|_\tau, \|z\|_\tau\right\},$$

where

$$\|y\|_\tau = \sup_{t\geq\tau}\left(\|T(t,\tau)y\|e^{-\overline{\lambda}(t-\tau)}\right) + \sup_{t<\tau}\left(\|T(t,\tau)y\|e^{-\underline{\lambda}(t-\tau)}\right)$$

and

$$\|z\|_\tau = \sup_{t<\tau}\left(\|T(t,\tau)z\|e^{\underline{\mu}(\tau-t)}\right) + \sup_{t\geq\tau}\left(\|T(t,\tau)z\|e^{\overline{\mu}(\tau-t)}\right).$$

One can easily verify that condition (2.2) holds. Moreover, for $t \geq \tau$ we have

$$\begin{aligned}\|T(t,\tau)y\|_t &= \sup_{s\geq t}\left(\|T(s,t)T(t,\tau)y\|e^{-\overline{\lambda}(s-\tau)}\right) + \sup_{s<t}\left(\|T(s,t)T(t,\tau)y\|e^{-\underline{\lambda}(s-t)}\right)\\ &\leq \sup_{s\geq t}\left(\|T(s,\tau)y\|e^{-\overline{\lambda}(s-t)}\right) + \sup_{\tau\leq s<t}\left(\|T(s,\tau)y\|e^{-\overline{\lambda}(s-t)}\right) + \sup_{s<\tau}\left(\|T(s,\tau)y\|e^{-\underline{\lambda}(s-t)}\right)\\ &\leq 2\sup_{s\geq\tau}\left(\|T(s,\tau)y\|e^{-\overline{\lambda}(s-t)}\right) + \sup_{s<\tau}\left(\|T(s,\tau)y\|e^{-\underline{\lambda}(s-t)}\right),\end{aligned}$$

where in the last inequality we have used that $\overline{\lambda} \geq \underline{\lambda}$. Hence,

$$\begin{aligned}\|T(t,\tau)P(\tau)x\|_t &\leq 2e^{\overline{\lambda}(t-\tau)}\sup_{s\geq\tau}\left(\|T(s,\tau)P(\tau)x\|e^{-\overline{\lambda}(s-\tau)}\right)\\ &\quad+e^{\underline{\lambda}(t-\tau)}\sup_{s<\tau}\left(\|T(s,\tau)P(\tau)x\|e^{-\underline{\lambda}(s-\tau)}\right)\\ &\leq 2e^{\overline{\lambda}(t-\tau)}\|x\|_\tau,\end{aligned}$$

again since $\overline{\lambda} \geq \underline{\lambda}$. Analogously, for $t \geq \tau$ we have

$$\begin{aligned}\|T(\tau,t)z\|_\tau &= \sup_{s\leq\tau}\left(\|T(s,\tau)T(\tau,t)z\|e^{\underline{\mu}(\tau-s)}\right) + \sup_{s>\tau}\left(\|T(s,\tau)T(\tau,t)z\|e^{\overline{\mu}(\tau-s)}\right)\\ &\leq \sup_{s\leq\tau}\left(\|T(s,t)z\|e^{\underline{\mu}(\tau-s)}\right) + \sup_{\tau<s\leq t}\left(\|T(s,t)z\|e^{\overline{\mu}(\tau-s)}\right) + \sup_{s>t}\left(\|T(s,t)z\|e^{\overline{\mu}(\tau-s)}\right)\\ &\leq 2\sup_{s\leq t}\left(\|T(s,t)z\|e^{\underline{\mu}(\tau-s)}\right) + \sup_{s>t}\left(\|T(s,t)z\|e^{\overline{\mu}(\tau-s)}\right),\end{aligned}$$

where in the last inequality we have used that $\underline{\mu} \leq \overline{\mu}$. Hence,

$$\begin{aligned}\|T(\tau,t)Q(t)x\|_\tau &\leq 2e^{\underline{\mu}(\tau-t)}\sup_{s\leq t}\left(\|T(s,t)Q(t)x\|e^{\underline{\mu}(t-s)}\right) + e^{\overline{\mu}(\tau-t)}\sup_{s>t}\left(\|T(s,t)Q(t)x\|e^{\overline{\mu}(t-s)}\right)\\ &\leq 2e^{-\underline{\mu}(t-\tau)}\|x\|_t,\end{aligned}$$

again since $\underline{\mu} \leq \overline{\mu}$. One can show in a similar manner that

$$\|T(t,\tau)P(\tau)x\|_t \leq e^{\underline{\lambda}(t-\tau)}\|x\|_\tau$$

and

$$\|T(\tau,t)Q(t)x\|_\tau \leq e^{-\overline{\mu}(t-\tau)}\|x\|_t$$

for $t \leq \tau$. Therefore, the evolution family $T(t,\tau)$ admits a strong exponential dichotomy with respect to the family of norms $\|\cdot\|_t$.

It remains to show that the map $t \mapsto \|x\|_t$ is measurable for each x. Let

$$g(\tau) = \sup_{t \geq \tau} \left(\|T(t,\tau)y\| e^{-\overline{\lambda}(t-\tau)} \right).$$

Since the function under the supremum is continuous, we have

$$g(\tau) = \sup_{t \in \mathbb{Q} \cap [\tau, +\infty)} \left(\|T(t,\tau)y\| e^{-\overline{\lambda}(n-\tau)} \right).$$

Now write $\mathbb{Q} = \{t_1, t_2, \dots\}$ and for each $n \in \mathbb{N}$ define

$$g_n(\tau) = \|T(t_n,\tau)y\| e^{-\overline{\lambda}(t_n-\tau)} \chi_{(-\infty, t_n]}(\tau).$$

The function g_n is measurable and hence, $g = \sup_n g_n$ is also measurable. One can show in a similar manner that the three other suprema in the definition of the norm are also measurable.

Conversely, assume that $T(t,\tau)$ admits a strong exponential dichotomy with respect to a family of norms satisfying (2.2) for some constants $C > 0$ and $\varepsilon \geq 0$. Then

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq \|T(t,\tau)P(\tau)x\|_t \\ &\leq D e^{\overline{\lambda}(t-\tau)} \|x\|_\tau \\ &\leq C D e^{\overline{\lambda}(t-\tau) + \varepsilon|\tau|} \|x\| \end{aligned}$$

and

$$\begin{aligned} \|T(\tau,t)Q(t)x\| &\leq \|T(\tau,t)Q(t)x\|_\tau \\ &\leq D e^{-\underline{\mu}(t-\tau)} \|x\|_t \\ &\leq C D e^{-\underline{\mu}(t-\tau) + \varepsilon|t|} \|x\| \end{aligned}$$

for $x \in X$ and $t \geq \tau$. Similarly,

$$\|T(t,\tau)Q(\tau)x\| \leq C D e^{\underline{\lambda}(\tau - t) + \varepsilon|\tau|} \|x\|$$

and

$$\|T(t,\tau)Q(\tau)x\| \leq C D e^{-\overline{\mu}(t-\tau) + \varepsilon|t|} \|x\|$$

for $x \in X$ and $t \leq \tau$. This shows that $T(t,\tau)$ admits a strong nonuniform exponential dichotomy. ☐

The following robustness result for the notion of a strong nonuniform exponential dichotomy is an immediate consequence of Theorem 5.1 and Proposition 6.1.

Theorem 6.2. *Assume that the evolution family $T(t,\tau)$ admits a strong nonuniform exponential dichotomy and that $B \colon \mathbb{R} \to B(X)$ is a strongly continuous function such that*

$$\|B(t)\| \leq c e^{-\varepsilon|t|}, \quad t \in \mathbb{R}.$$

If c is sufficiently small, then the evolution family $U(t,\tau)$ defined by

$$U(t,\tau)x = T(t,\tau)x + \int_\tau^t T(t,s)B(s)U(s,\tau)x \, ds, \quad t,\tau \in \mathbb{R}$$

admits a strong nonuniform exponential dichotomy.

Acknowledgement

Partially supported by FCT/Portugal through UID/MAT/04459/2013.

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